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Hilbert transforms using fast Fourier transforms

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Abstract. Fourier transforms are unreliable near discontinuities because of the Gibbs phenomenon. Before using the fast Fourier transform technique to evaluate Hilbert transforms, it is desirable to remove any discontinuities by smoothing, as suggested by Papoulis (1962). This is especially true if we wish to use Fourier transforms to find the Hilbert transform $h(t)$ of a function $f(t)$ which has an infinite discontinuity: it is then necessary to smooth $f(t)$ as well as the Hilbert transform kernel. An alternative to smoothing $f(t)$ is to remove a discontinuity at a point $t = d$ by multiplying $f(t)$ by the factor $(t-d)$: this has the advantage of having better asymptotic behaviour. A numerical example is given and here the two methods perform about equally.

1. Introduction

Hilbert transforms occur frequently in mathematical physics—dispersion relations such as the Kramers-Kronig equations of dielectric theory and the solution of Laplace's equation in a half-plane are two examples. Yet there are few techniques for their numerical evaluation. One technique, which has been suggested several times since about 1970, is to use the fast Fourier transform (FFT). However, early papers on the application of FFT methods to numerical Hilbert transforms paid insufficient attention to the problems of wrap-around and discontinuities in the time and frequency domains, and it is the purpose of this paper to discuss some ways for resolving these problems in a few simple cases.

In § 2 we describe the basic FFT method for evaluating convolutions and set the notation for Hilbert and Fourier transforms which are used later.

In using FFTs to find an approximation to the Hilbert transform $h(t)$ of a given function $f(t)$, the overall procedure consists of nine main steps: (i) trucating $f(t)$ if necessary; (ii) smoothing $f(t)$ to avoid singularities; (iii) discretising the smoothed $f(t)$; (iv) discrete Fourier transform of the sampled and smoothed $f(t)$; (v) truncation of the Hilbert kernel in the time domain; (vi) smoothing the kernel; (vii) discrete Fourier transform of the truncated kernel; (viii) multiplication of the Fourier transforms; (ix) inverse discrete Fourier transform.

The order in which these operations are applied is important, as indeed is the precise manner of their implementation, but some variation, with slight difference in interpretation, is possible. The equivalence of these variations follows from the fact that convolutions commute and from the Fourier transform property of convolutions. For example, it is possible to replace step (vi) with a multiplication by a window function at step (viii), or, equivalently, stage (vi) could be omitted if, after stage (iii), the sampled function is suitably interpolated. Since there are many possible permutations in these steps, they are not discussed in detail. It is only necessary to stress that,

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if $f(t)$ is smoothed as described in § 3, for example, this should be done before sampling and we discuss this point in § 4.

Smoothing of the Hilbert kernel is also necessary, even when the function $f(t)$ is well behaved, and our version of this is set out in § 5.

Since the main difficulty in applying the Fourier transform method occurs when the functions involved have discontinuities, giving very slow convergence, we will give an alternative suggestion for dealing with discontinuities of a particular type in § 6.

2. Hilbert transform

The Hilbert transform $h(t)$ of the function $f(t)$ is the principal value of the convolution of $f(t)$ with the kernel function $k(t) = -1/\pi t$:

$$h(t) = \int_{-\infty}^{\infty} f(s)\, k(t-s)\, \mathrm{d}s = (1/\pi) \int_{-\infty}^{\infty} f(s)(t-s)^{-1}\, \mathrm{d}s. \tag{2.1}$$

Using $*$ to denote convolution we may write this as

$$h(t) = f(t) * k(t).$$

Fourier transforms will be denoted by the corresponding capital letter so that, for example,

$$F(\omega) = \int_{-\infty}^{\infty} f(x) \exp(\mathrm{i}\omega x)\, \mathrm{d}x. \tag{2.2}$$

The Hilbert transform relation (2.1) becomes, on Fourier transformation,

$$H(\omega) = F(\omega)K(\omega) = -\mathrm{i}\, \mathrm{sgn}(\omega)\, F(\omega). \tag{2.3}$$

Whereas the direct calculation of the Hilbert transform proceeds from (2.1) regarded as an integral to be evaluated at a fixed argument t, for example Taurian (1980) uses cubic splines in this way, the Fourier transform technique proceeds by finding $H(\omega)$ for all ω and then taking the inverse Fourier transform.

The simplest FFT procedure is to discretise and truncate $f(t)$ so that (2.2) becomes a discrete Fourier transform. Then (2.3) is used to get $H(\omega)$ and an inverse discrete Fourier transform gives $h(t)$ at the sampling points. This was the procedure used by Peterson and Knight (1970), Henery (1970), Saxton (1974), and others, but it is much affected by wrap-around effects. To avoid wrap-around, the functions $f(t)$ and $k(t)$ must be truncated and (2.1) evaluated as a convolution using the methods of Cooley *et al* (1967) or Nussbaumer (1981). Even then there are problems with the singularity of the kernel $k(t)$ at the origin necessitating a smoothing operation.

Discontinuities may also occur in $f(t)$. We take an example from Kikuchi and Fukao (1976) concerning dislocation models of stress in earthquakes. The stress field $h(t)$ is the Hilbert transform of the derivative $g'(t)$ of the dislocation displacement $g(t)$, and in the models considered by Kikuchi and Fukao (1976) the derivative $g'(t)$ may be infinite at the ends of the dislocation. However, we may write (2.1) as

$$h(t) = \int_{-\infty}^{\infty} g'(s)\, k(t-s)\, \mathrm{d}s = \int_{-\infty}^{\infty} k(t-s)\, \mathrm{d}g(s) \tag{2.4}$$

and the last form of the integral is preferable in this case, as we now show.

3. Smoothing $f(t)$

In numerical work there must be an upper bound on the frequencies included in the Fourier transform, so that the Gibbs phenomenon will give rise to difficulties near any discontinuity. To deal with discontinuities, Papoulis (1962) uses the following procedure based on the running average $\bar{f}(t)$:

$$\bar{f}(t) = (1/\Delta) \int_{t-\Delta/2}^{t+\Delta/2} f(s)\, ds = (1/\Delta)[g(t+\Delta/2) - g(t-\Delta/2)].$$

Although other choices of average are possible, this choice of average is appropriate for (2.4) as it leads to a simple form of product integration. The function $\bar{f}(t)$ is continuous and equals the convolution of $f(t)$ with the rectangular function $(1/\Delta)r(2t/\Delta)$, where $r(t) = 1$ if $|t| < 1$, and $r(t) = 0$ if $|t| > 1$. If the Fourier transform $\bar{F}(\omega)$ is now truncated above $\omega = \Omega$, the transform $\bar{F}_\Omega(\omega)$ results:

$$\begin{aligned}\bar{F}_\Omega(\omega) &= \bar{F}(\omega) r(\omega/\Omega) \\ &= F(\omega)\, \mathrm{sinc}(\omega\Delta/2) r(\omega/\Omega),\end{aligned}$$

and the inverse function will give a satisfactory approximation to $f(t)$ without overshoot.

For numerical work using digital computers it is also necessary to sample the function $f(t)$ at the points $t = m\Delta$, $m = 0, -1, +2, \ldots$. The most appropriate choice of Ω is therefore $\Omega = \pi/\Delta$, by virtue of the sampling theorem.

Since convolutions commute, the Hilbert transform $\bar{h}(t)$ of $\bar{f}(t)$ is a smoothed version of $h(t)$:

$$\begin{aligned}\bar{h}(t) &= k(t) * \bar{f}(t) = k(t) * (1/\Delta) r(2t/\Delta) * f(t) \\ &= (1/\Delta) r(2t/\Delta) * h(t).\end{aligned}$$

This allows us to assess the effect of smoothing on $h(t)$. For example, if $h(t)$ has a continuous third derivative in the interval $(t-\Delta/2, t+\Delta/2)$, the difference between $h(t)$ and $\bar{h}(t)$ is $h''(t)\Delta^2/24 + O(\Delta^3)$ as is easily shown by integrating the Taylor series expansion for $h(t)$ over that interval. More generally, if an even smoothing operator, localised to a range Δ, is applied to $f(t)$, then the error in $h(t)$ will be of order $h''(t)\Delta^2$, when $h(t)$ has a continuous third derivative over the range Δ.

However, the main point is to apply the smoothing operator before sampling $f(t)$ since these operations do not commute unless $f(t)$ is band-limited.

4. Sampling $f(t)$

We can describe the effect of sampling most easily using a train of impulses $s(t) = \Delta \sum_{n=-\infty}^{\infty} \delta(t - n\Delta)$, where $\delta(t)$ is the Dirac delta function. The sampled function $f_s(t)$ may be written formally as a succession of impulses:

$$f_s(t) = \Delta \sum_{n=-\infty}^{\infty} f(t)\delta(t - n\Delta) = f(t)s(t),$$

and its transform, which has period $2\pi/\Delta$ in ω, is

$$F_s(\omega) = \sum_{n=-\infty}^{\infty} F(\omega - 2\pi n/\Delta). \tag{4.1}$$

Because of aliasing, every frequency ω is confounded with an infinite number of higher frequencies and this causes severe problems if $F(\omega)$ has a long tail. However, if $F(\omega)$ is negligible for $\omega > \pi/\Delta$, as will be the case if $f(t)$ is a smooth function, $F_s(\omega)$ will represent accurately the low frequency spectrum of $f(t)$. This implies that $f(t)$ should be smoothed before being sampled, and with this understanding we can write $\bar{f}_s(t)$ for the result of sampling $\bar{f}(t)$:

$$\bar{f}_s(t) = \bar{f}(t)s(t). \tag{4.2}$$

The spectrum $\bar{F}_s(\omega)$ is related to $F(\omega)$ by

$$\bar{F}_s(\omega) = \sum_{n=-\infty}^{\infty} F(\omega - 2\pi n/\Delta)\,\mathrm{sinc}\,[\tfrac{1}{2}\Delta(\omega - 2\pi n/\Delta)]. \tag{4.3}$$

The reduction in aliasing of (4.3) compared to (4.1) is most marked for small ω, say for $0 < \omega \ll \pi/\Delta$, in which case the coefficient of $F(\omega)$ in (4.3) is approximately $1 - \omega^2\Delta^2/24$, whereas the coefficient of $F(\omega - 2n\pi/\Delta)$, $n \neq 0$, is of order $\omega\Delta/2n\pi$. To this approximation $\bar{F}_s(\omega)$ is equal to

$$F(\omega)(1 - \omega^2\Delta^2/24) + (\omega\Delta/2\pi) \sum_{n \neq 0} (1/n)F(\omega - 2n\pi/\Delta). \tag{4.4}$$

To the same approximation, sampling $f(t)$ and then smoothing would give, in place of (4.4), the expression (4.5):

$$\left(F(\omega) + \sum_{n \neq 0} F(\omega - 2n\pi/\Delta) \right)(1 - \omega^2\Delta^2/24). \tag{4.5}$$

Comparison of (4.4) and (4.5) shows that aliasing is less in (4.4) by a factor of order $\omega\Delta/2\pi$ when $\omega\Delta$ is small. The bias term $-\omega^2\Delta^2 F(\omega)/24$ is common to both and may be identified with $h''(t)\Delta^2/24$ by noting that $f''(t)$ has Fourier transform $-\omega^2 F(\omega)$ and that $h''(t)$ is the Hilbert transform of $f''(t)$.

Of course the difference between (4.4) and (4.5) is of importance only if $F(\omega)$ has a long tail, as will be the case when $f(t)$ has discontinuities, and in this case it is preferable to smooth before sampling. On the other hand, if $F(\omega)$ is band-limited, $F(\omega) = 0$ for $|\omega| > \pi/\Delta$, smoothing is not desirable as the sampled $f_s(t)$ contains all the information present in $f(t)$. The use of smoothing would then introduce bias with no compensating improvement in aliasing.

5. Smoothed $k(t)$

The infinite convolution in the Hilbert transform (2.1) must be truncated in practice, so we must assume that $f(t)$ is negligible outside some interval $(-d, d)$. It is then easy to see that the evaluation of $h(t)$ involves values of the kernel $k(s)$ in the interval $(t-d, t+d)$. If we are interested only in the values of $h(t)$ in some interval $(-t_0, t_0)$, the kernel $k(s)$ need only be defined in the interval $(-t_0, -d, +t_0 + d)$, so we may truncate the kernel $k(s)$ above $T = t_0 + d$. The effective kernel is now $k_T(s) = k(s)r(s/T)$, and the convolution to be evaluated is now

$$h_T(s) = \int_{-\infty}^{\infty} f(u)k_T(s-u)\,\mathrm{d}u \tag{5.1}$$

and by construction this is equal to $h(s)$ in the interval $(d - T, T - d)$. Note also that

$h_T(s) = 0$ for $|s| > T$, so that wrap-around, which is inherent in FFT methods, is thus avoided.

The Fourier transform of the truncated kernel is

$$K_T(\omega) = -(2i/\pi)\,\mathrm{Si}(\omega T), \tag{5.2}$$

and so the Fourier transform of $h_T(s)$ will be calculated from (5.3) rather than (2.3):

$$H_T(\omega) = F(\omega)K_T(\omega). \tag{5.3}$$

Clearly if $F(\omega)$ is continuous we recover (2.3) in the limit $T \to \infty$.

The numerical procedure will use the sampled function $f_s(t)$ rather than $f(t)$ in (5.1), with a consequent change in (5.3). However, to anticipate matters a little, we may note that (5.3), and therefore (5.2), need only be evaluated for $\omega T = m\pi$, m integer, and a convenient formula for $\mathrm{Si}(x)$ is given in Abramowitz and Stegun (1967).

We consider now the Hilbert transform of $f_s(t)$, which is a sequence of impulses. Even if $f(t)$ is band-limited some kind of smoothing, or some kind of integration rule, will still be necessary as the Hilbert transform of a succession of impulses contains discontinuities centred on the sampling points. To keep the argument general, suppose that an interpolation operation, in the form of a convolution with a weight function $w(t)$, is applied to $f_s(t)$ and the result, denoted by $f_{ws}(t)$ is then Hilbert transformed. Since convolutions commute, the Hilbert transform $h_{ws}(t) = k(t) * w(t) * f_s(t)$ can be regarded as a transform with effective kernel $k_w(t) = k(t) * w(t)$. For example, this is the approach of Liu and Kosloff (1981), who use for $w(t)$ the triangular function $(1/\Delta)v(t/\Delta)$. where $v(t) = 1 - |t|$, $|t| \leqslant 1$, $v(t) = 0$, $|t| > 1$.

The effective kernel $k_w(t)$ may then be truncated above $T = N\Delta/2$ and sampled at $t = m\Delta$, $m = 0, \pm 1, \pm 2, \ldots, \pm(N/2 - 1)$, so that the output Hilbert transform is a discrete convolution to which the FFT method may be applied. The result is that (5.3) is replaced by

$$H_{sw}(\omega) = F_s(\omega)K_{wT}(\omega) \tag{5.4}$$

where $K_{wT}(\omega)$ is the discrete Fourier transform of the discretised truncated kernel $k_{wt}(t) = k_w(t)r(t/T)s(t)$. Once $K_{wT}(\omega)$ is found it may be written

$$K_{wT}(\omega) = K_T(\omega)\,W(\omega)$$

where $W(\omega)$ is a spectral window function which is defined by $w(x)$. A more direct attack is to choose $W(\omega)$ as a window function in the first place, thus avoiding the calculation of a Fourier transform. For the choice $W(\omega) = \cos(\omega\Delta/2)$ for $|\omega| \leqslant \pi/\Delta$ and $W(\omega) = 0$ for $|\omega| > \pi/\Delta$, equation (5.4) becomes

$$H_{sw}(\omega) = F_s(\omega)K_T(\omega)\cos(\omega\Delta/2), \qquad |\omega| \geqslant \pi/\Delta. \tag{5.5}$$

If the discrete Fourier transform $F_s(\omega)$ is given for $\omega = 2n\pi/N\Delta$, (5.5) will give the discrete Fourier transform $H_{sw}(\omega)$ which may be inverted to give the Hilbert transform $h_{sw}(t)$ for $t = n\Delta$. The method based on (5.5) will be called the 'smoothed kernel' method.

Smoothing both $f(t)$ and $k(t)$ means using $\bar{F}_s(\omega)$ rather than $F_s(\omega)$ in (5.5) giving the '$f(t)$ and $k(t)$ smoothed' method:

$$\bar{H}_{sw}(\omega) = \bar{F}_s(\omega)K_T(\omega)\cos(\omega\Delta/2), \qquad |\omega| \leqslant \pi/\Delta. \tag{5.6}$$

Finally, we note that a smoothed kernel of a type similar to (5.2) was considered by Saito (1974) and Tretter (1976). Saito (1974) proposed a discrete Hilbert transform

based on an effective kernel $k_s(t) = -(1/\pi t)(1-\cos \pi t/\Delta)$ with Fourier transform $K_s(\omega) = -i \operatorname{sgn}(\omega) r(\omega\Delta/\pi)$. The same discrete filter was suggested by Tretter (1976) for the Hilbert transform of a band-limited function. If $f_s(t)$ is passed through an ideal band-pass filter the result is well known to be the Whittaker interpolation of $f_s(t)$, and if this interpolated function is then Hilbert transformed, the overall filter has an effective kernel $k_s(t)$.

6. Discontinuity factor

It will often be the case that $f(t)$ cannot be integrated directly, so that Papoulis' (1962) smoothing method cannot be used. An alternative which is suitable for sampled functions, and which has the added advantage of having better long time behaviour, will now be presented.

We assume now that $f(t)$ can be written $f(t) = (d^2 - t^2)^{-1}\phi(t)$, where $\phi(t)$ is continuous for all t. For the numerical example used in § 7, $f(t) = t(d^2 - t^2)^{-1/2}$, $|t| < d$; $f(t) = 0$, $|t| > d$.

Let $h(t)$ be the Hilbert transform of $f(t)$, designated symbolically as $f(t) \to h(t)$. From elementary results in Hilbert transform theory, (see for example Sneddon (1972)), the transform of $(t+d)f(t)$ is given via (6.1):

$$(t+d)f(t) \to (t+d)h(t) + (1/\pi)\int_{-\infty}^{\infty} f(y)\,dy, \tag{6.1}$$

and repeating the process with a factor $(t-d)$ gives

$$(t^2-d^2)f(t) \to (t^2-d^2)h(t) + (1/\pi)\int_{-\infty}^{\infty} (t+y)f(y)\,dy. \tag{6.2}$$

When the integrals $\mu_i = (1/\pi)\int_{-\infty}^{\infty} y^i f(y)\,dy$ are known or can be evaluated to high accuracy, we can use (6.2) to find $h(t)$. If the Hilbert transform of $(t^2-d^2)d(t)$, which has no singularities by assumption, is estimated numerically to be $h^{\#}(t)$ say, then $h(t)$ is given by

$$h(t) = [h^{\#}(t) - t\mu_0 - \mu_1]/(t^2-d^2). \tag{6.3}$$

One disadvantage in using (6.3) is that even small numerical errors in μ_0, μ_1 or $h^{\#}(t)$ will lead to large errors in $h(t)$ near $|t| = d$. One advantage, apart from avoiding the dicontinuity, is that wrap-around effects are diminished at large t. This can be shown easily using the asymptotic expansion given by Taurian (1980):

$$h(t) = -\sum_{n=0}^{\infty} \mu_n / t^{n+1}. \tag{6.4}$$

For sufficiently large t this expansion gives a useful estimate of $h(t)$ provided $f(t)$ is concentrated in a finite interval. In many applications $f(t)$ is either even or odd, and in these cases the expansion (6.4) loses every second term. Note also that the term $-(t\mu_0+\mu_1)/(t^2-d^2)$ in (6.3) contains the first two terms of the expansion (6.4), implying that $h^{\#}(t)$ tends to zero as $(\mu_2 - \mu_1 d^2)/t$ and is therefore negligible compared to $t\mu_0+\mu_1$ (assuming one of μ_0 and μ_1 to be non-zero). For this reason wrap-around effects are not quite so serious in the estimation of $h^{\#}(t)$ as they are for $h(t)$.

7. Numerical example

The five methods to be compared are referred to as: no smoothing; $f(t)$ smoothed; $k(t)$ smoothed; $f(t)$ and $k(t)$ smoothed; discontinuity factor. These are based, respectively, on equations: (5.3); (5.4); (5.5); (5.6); and (6.3) with (5.5).

The specific example used to test these methods comes from Kikuchi and Fukao (1976), the same example is used by Liu and Kosloff (1981).

Example. For the function given by

$$f(t) = t(d^2 - t^2)^{-1/2}, \qquad |t| < d$$
$$= 0, \qquad |t| > d$$

the Hilbert transform is

$$h(t) = 1, \qquad |t| < d$$
$$= 1 - |t|(t^2 - d^2)^{-1/2}, \qquad |t| > d.$$

The parameters used throughout were: $\Delta = 0.01$; $N = 1024$; $d = 2.0$. With this choice of parameter the point $t = 2.0$, an infinite discontinuity in both $f(t)$ and $h(t)$, lies exactly on the 200th sampling point, and this is rather awkward for methods 1 and 3. In these cases we simply set $f(2) = f(2^+)$, but it is clear that a slightly different choice of Δ may give very different results.

In this example, $f(t)$ is already truncated at $t = 2.0$: $k(t)$ is truncated at $N\Delta/2 = 5.12$, so that wrap-around will affect the estimated $h(t)$ for $t > 3.12$. With this in mind, we looked at the largest absolute errors in the estimates of $h(t)$ within five intervals of t (excluding the point of discontinuity $t = 2.0$). These are shown in table 1 for each of the five methods. In every case the largest absolute error overall occurred at the sampling points immediately adjacent to the discontinuity, i.e. at $t = 2.0 \pm 0.01$.

Although we cannot draw too many conclusions from just one example, it is clear from table 1 that it is possible to use FFT methods for Hilbert transforms, but only if discontinuities are suitably dealt with. Smoothing $f(t)$, if necessary, must be applied before sampling; $k(t)$ must always be smoothed. Method 5, using the discontinuity factor, is a useful alternative to smoothing $f(t)$, and has the added advantage of smaller wrap-around effects.

Table 1. The maximum errors for five approximate Hilbert transforms in various ranges. For any given interval, the largest error is usually at the end of the interval nearest to the discontinuity. The discontinuity $t = 2$ is not included. Wrap-around effects are responsible for the large errors in the interval $3 < t \leqslant 4$.

	Maximum error for t in interval				
Method	$0 \leqslant t < 1$	$1 \leqslant t < 1.8$	$t = 2 \pm 0.01$	$2.1 < t \leqslant 3$	$3 < t \leqslant 4$
No smoothing	0.0847	0.3317	8.613	0.5686	0.1854
$f(t)$ smoothed	0.0293	0.1173	3.818	0.2370	0.2217
$k(t)$ smoothed	0.0616	0.2351	7.890	0.4042	0.1754
$f(t)$ and $k(t)$ smoothed	0.0001	0.0009	2.132	0.0061	0.2096
Discontinuity factor	0.0001	0.0017	0.957	0.0045	0.0158

If we are content to calculate $h(t)$ over the interval $(-3.12, 3.12)$ we could improve the accuracy of table 1, by a factor of 64 approximately, by taking N to be $8 * 1024$ and $\Delta = 0.01/8$.

A more successful approach when $f(t)$ has a continuous fourth derivative is to use cubic splines as in Taurian (1980). The Hilbert transform is then given by an explicit formula in terms of the coefficients of the approximating polynomials. This has the added advantage that the transform may be evaluated for complex argument. The order of convergence is Δ^4 rather than Δ^2 as for the smoothing method suggested above or for Liu and Kosloff's (1981) linear interpolation method.

For the sake of comparison we note that Taurian (1980) also suggested that linear interpolation might be used rather than cubic splines, which would be exactly equivalent to Liu and Kosloff's (1981) method but for the choice of points at which to evaluate the transform.

The methods outlined above have been tested on other numerical examples with discontinuities. The results bear out the general conclusion that, of the methods described, only those methods which explicitly allow for the discontinuity, either by appropriate smoothing or by the method of § 6, give acceptable error control away from the discontinuity.

8. Final remarks

The general rule must be to try to do most of the work analytically, if possible, before falling back on purely numerical techniques. This may mean constructing some exactly transformable approximating function and using the Fourier transform technique on the difference. Thus, if there are points of discontinuity in the transformed function, no finite Fourier transform will represent the function adequately in the vicinity of the discontinuity (this is the Gibbs phenomenon), and the discontinuity should be removed before embarking on the Fourier transform method. We may illustrate the general idea with the example of an even function which has a finite jump discontinuity of ε at $t = d$. By adding or subtracting the rectangular function $\varepsilon r(t/d)$, where $r(t) = 1$ for $|t| < 1$ and $r(t) = 0$ elsewhere, we may reduce the numerical work to that of transforming a continuous function of t.

In the same vein it may be possible to use a polynomial approximation for a function $f(t)$ continuous over a finite interval. The polynomial might be chosen so that its first q moments coincide with those of $f(t)$, ensuring that its Hilbert transform, which of course may be exactly computed, has the right asymptotic behaviour. The author has used Bernstein polynomials for this purpose—largely because they are defined by the sample values at evenly spaced ordinates—but the convergence to the proper transform is so slow that large-order polynomials have to be used and then there are problems with floating point accuracy for large t.

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